Postulates of Standard Quantum Mechanics

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Lecture 1 Jan 4, 2005

Postulate I. A physical configuration (state) is described by a state operator that is non-negative (Hermitian) operator with unit trace. Rank-one projectors, $\hat{\rho} = |\psi\rangle\langle\psi|$, called pure states, correspond to states of maximal knowledge.

In many applications, it is adequate to specify the quantum state using only vectors $|\psi\rangle$, where these vectors are elements of a Hilbert space. A Hilbert space $\mathcal{H}$ is a linear vector space with an inner product defined on it, $(\psi, \phi) \in \mathbb{C}$, or in Dirac notation, $\langle\psi|\phi\rangle \in \mathbb{C}$. (We will see later that for a vector space to qualify as an infinite dimensional Hilbert space we must specify a further condition.)

The dimension of $\mathcal{H}$ is the maximum number of linearly independent vectors.

A linearly independent set of vectors spanning $\mathcal{H}$ is called a basis.

Any vector can be expressed as a linear combination of basis vectors, e.g., let $\{\phi_j\}$ be a basis of $\mathcal{H}$, $|\psi\rangle \in \mathcal{H}$, then $|\psi\rangle = \sum_j c_j|\phi_j\rangle$.

Example 1. *A linearly independent set of column vectors form a basis for a discrete Hilbert space.*

Example 2. *The space of differentiable functions can form a Hilbert space.*

An *inner product* is defined by the properties:

i) $(\psi, \phi) \in \mathbb{C}$
ii) $(\phi, \psi) = (\psi, \phi)^\star$ ($\star$ denotes complex conjugation)
iii) $(\phi, c_1\psi_1 + c_2\psi_2) = c_1(\phi, \psi_1) + c_2(\phi, \psi_2)$
iv) $||\psi||^2 = (\psi, \psi) >= 0$

In Dirac's notation (i) takes the form: $\langle\psi|\phi\rangle \in \mathbb{C}$

An orthonormal basis $\{\phi_j\}$ has

$$(\phi_j, \phi_i) = \langle\phi_j|\phi_i\rangle = \delta_{ij} \tag{1}$$

where δ_{ij} is the Kronecker delta-function.

Example 3. *For column vectors with elements,*

$$|\phi\rangle \equiv \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

$$|\psi\rangle \equiv \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

the inner product is expressed as follows:

$$\langle\psi|\phi\rangle = \sum_j b_j^\star a_j$$

Note that bra vectors (e.g. $\langle\psi|$) are elements of a dual space $\mathcal{H}^\dagger$, which consists of linear functionals mapping elements of the Hilbert space to complex scalars.

Example 4. *Let $\psi(x)$ and $\phi(x)$ be complex functions, then the inner product takes the form:* $\langle\psi|\phi\rangle = \int d\mu(x)\psi^\star(x)\phi(x)$

An infinite dimensional $\mathcal{H}$ has to be complete in the norm – that is, all vectors obtained from limits of Cauchy sequences are contained in $\mathcal{H}$. Given a Cauchy sequence $\{\psi_m\}$, $||\psi_m\rangle - |\psi_n\rangle| \to 0$ as $m, n \to \infty$, $|\psi\rangle = lim_{m\to\infty}|\psi_m\rangle \in \mathcal{H}$, and $\|\psi\|^2 < \infty$.

An important case of a Hilbert space is $L^2(a, b)$, that is, the set of square integrable complex functions, or

$$\int_a^b dx|\psi(x)|^2 < \infty. \tag{2}$$

In practice it is convenient to make use of non-square integrable and generalized functions which do not fit in the Hilbert space framework, for example,

$$\psi(x) = \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-i\frac{px}{\hbar}\right). \tag{3}$$

and the Dirac 'delta-function':

$$\delta(x - x_o) = \langle x|x_o\rangle, \tag{4}$$

defined by the conditions,

$$\begin{aligned}\int dx\delta(x - x_o) &= 1 \\ \int dx\delta(x - x_o)f(x) &= f(x_o)\end{aligned}$$

To accommodate these elements we can use the *rigged Hilbert space* formalism and treat the following inner products as well-defined:

$$\langle x|x_0\rangle = \delta(x - x_0) \tag{5}$$
$$\langle p|p_0\rangle = \delta(p - p_0) \tag{6}$$

A state operator $\hat\rho$ must be non-negative. An operator is non-negative iff $\langle\mu|\hat\rho|\mu\rangle \geq 0$ for all $|\mu\rangle \in \mathcal{H}$.

Note that this guarantees that the state operator $\hat A$ is Hermitian ($\hat A = \hat A^\dagger$), where the adjoint operation will be defined later.

For a normalized state operator $\hat\rho$ ($\mathrm{tr}(\hat\rho) = 1$) there are three equivalent definition of purity (maximal knowledge):

1) $\hat\rho^2 = \hat\rho$, which means that ρ is projector.
2) $\mathrm{tr}(\hat\rho^2) = 1$.
3) $\hat\rho = |\psi\langle\rangle\psi|$, defining a projector onto a one-dimensional subspace of $\mathcal{H}$.

The *trace* of an operator $\hat{A}$ is defined by

$$\mathrm{tr}(\hat{A}) = \sum_j \langle \phi_j | \hat{A} | \phi_j \rangle \tag{7}$$

where $\{|\phi_j\rangle\}$ is any (convenient) normalized orthogonal basis.

Postulate II. Each physical observable is represented by a Hermitian operator $\hat{O}$. Let $\hat{O}$ be a Hermitian operator with eigenvalues λ_l and eigenvectors $|\lambda_l\rangle$.

a) **The set of possible observable outcomes is determined from $\{\lambda_l\}$.**
b) **The probability of outcome λ_l, is given by $\mathrm{Pr}(\lambda_l) = \mathrm{tr}(|\lambda_l\rangle\langle\lambda_l|\hat{\rho})$ when the physical configuration is described by $\hat{\rho}$.**

Postulate 2.a) is responsible for the novel structural aspects of quantum theory. Operators with discrete spectra are "quantized" (in the sense that they are discretized). Examples of this are the atomic energy levels, angular momentum, and electromagnetic radiation can only exchange discrete amounts of energy with some systems (i.e. "photons").

Postulate 2.b) provides the statistical/probabilistic/indeterministic character of quantum predictions. It is known as **the Born rule**. **Lecture 2** Jan 6, 2005

Hermitian operators are linear operators, i.e. $\hat{O}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1(\hat{O}|\psi_1\rangle) + c_2(\hat{O}|\psi_2\rangle)$, that are self-adjoint $\hat{A}^\dagger = \hat{A}$. The adjoint operator $\hat{A}^\dagger$ is defined by the condition $(\hat{A}^\dagger\phi, \psi) = (\phi, A\psi)$ for all $\psi, \phi \in \mathcal{H}$. (In Dirac notation this definition is expressed as: $\langle\phi|\hat{A}^\dagger|\psi\rangle = (\langle\psi|\hat{A}|\phi\rangle)^*$.)

Of course operators do not necessarily commute, i.e., it maybe that $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$, where we have just defined the *commutator* of operators A and B.

Hermitian operators are always diagonalizable, with eigenvectors and eigenvalues defined by the condition, $\hat{O}|\lambda_l\rangle = \lambda_l|\lambda_l\rangle$.

An important representation of a Hermitian operator is its *spectral decomposition*. In the case of a discrete spectrum this is,

$$\hat{O} = \sum_l \lambda_l |\lambda_l\rangle\langle\lambda_l|$$

where the projectors are $P_l = |\lambda_l\rangle\langle\lambda_l|$. For a continuous spectrum this is,

$$\hat{O} = \int d\lambda \lambda |\lambda\rangle\langle\lambda|$$

where $|\lambda\rangle$ may not be a Hilbert space vector but an element of a rigged Hilbert space (following Dirac's (1930) approach). An alternate approach is the generalized spectral decomposition proposed by von Neumann (1932).

The spectral decomposition allows us to define unambiguously a function of an operator

$$f(\hat{O}) = \sum_\lambda f(\lambda)|\lambda\rangle\langle\lambda|. \tag{8}$$

We can also compute expectation of an observable given by an operator via

$$\langle\hat{O}\rangle = \int d\lambda \langle\psi|\hat{O}|\psi\rangle = \mathrm{tr}(\hat{O}|\psi\rangle\langle\psi|). \tag{9}$$

The spectral decomposition is useful also for calculating probabilities according to the Born rule,

$$\mathrm{Pr}(\lambda \in [a, b]) = \mathrm{tr}\left[\int_a^b d\lambda |\lambda\rangle\langle\lambda||\psi\rangle\langle\psi|\right]. \tag{10}$$

Another useful result is the *(spectral) resolution of the identity*,

$$\hat{1} = \sum_l |\lambda_l\rangle\langle\lambda_l|. \tag{11}$$

Postulate III.

Dynamical transformations are generated by unitary operators,

$$\begin{aligned} |\psi(s_2)\rangle &= U(s_2, s_2)|\psi(s_1)\rangle & (12) \\ \hat{\rho}(s_2) &= U(s_2, s_1)\hat{\rho}(s_1)U^\dagger(s_2, s_1) & (13) \end{aligned}$$

By definition, an operator $\hat{U}$ is unitary if $\hat{U}\hat{U}^\dagger = \hat{1}$, which guarantees it can always be expressed in the form $\hat{U}(s) = \exp(is\hat{A})$, where $\hat{A}$ is a Hermitian operator.

Historically, Dirac identified the operator algebras associated with position, momentum and angular momentum by quantizing the classical Poisson bracket. This is not an ideal approach since we want quantum mechanics to be an independent theory based on its own set of postulates, and not dependent on a classical theory for its formulation.

T. Jordan proposed a different approach (Jordan, 1975) where the unitary transformation and the associated algebra of the generators $\{\hat{A}\}$ are identified from the considering the properties of the continuous Galilean space-time symmetries (generating displacements in position, rotations, and time-translations). Thus one is able to deduce that

$$\begin{aligned} \text{position translation} &\rightarrow \exp(-ia\hat{p}) \rightarrow [\hat{x}, \hat{p}] = iC & (14) \\ \text{rotation about } \hat{n} &\rightarrow \exp(-i\theta\hat{n}\cdot\hat{J}) \rightarrow [J_i, J_j] = iCJ_k = -[J_j, J_i] & (15) \end{aligned}$$

where C is an unspecified constant, that needs to be determined experimentally – this is the fundamental constant we call $\hbar$. From the time translation we deduce *Schrödinger's equation*,

$$\begin{aligned} |\psi(t_2)\rangle &= \hat{U}(t_2, t_1)|\psi(t_1)\rangle & (16) \\ \hat{H} &= i\hbar\left(\frac{d}{dt}\hat{U}\right)U^\dagger & (17) \\ \hat{H} &= -i\hbar\hat{U}\left(\frac{d}{dt}\hat{U}\right)^\dagger & (18) \\ \text{thus} \quad i\hbar\frac{d}{dt}|\psi(t)\rangle &= \hat{H}|\psi(t)\rangle. & (19) \end{aligned}$$

See Peres (1995) and Ballentine (1998).

We can view transformations also in the Heisenberg picture, where observables evolve instead of the state, $\hat{A}(t_2) = \hat{U}^\dagger(t_2, t_1)\hat{A}(t_1)\hat{U}(t_2, t_1)$.

Uncertainty Principle

The *uncertainty principle* due to Heisenberg (1925) states that at best

$$\delta x\delta p \simeq \hbar$$

where δx denotes the resolution for determining x for an individual system in a single experimental trial. There is, in principle, no limitation to how small the resolution can be for determining either x or p, but not both in the same setup. This result is motivated by *the Heisenberg microscope* which shows that simultaneous determination of position and momentum for a single particle is limited because of Einstein's relation $E = h\nu = \hbar\omega$ for the light as probe. The wave-particle duality for the light quanta places a limit on the resolution-disturbance tradeoff, as opposed to the classical picture where the energy of the light probe is proportional to intensity of the wave and there is no limit on the trade-off.

Heisenberg's uncertainty principle is often confused with Robertson's uncertainty principle. Define,

$$\Delta\hat{A}^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$$

to be the variance over an ensemble of measurement outcomes for the observable A. Then we can prove the generalized uncertainty inequality (Robertson, Phys. Rev 34, 163, 1929),

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|,$$

in which the uncertainty is non-zero if the two observables do not commute. In the case of position and momentum, this means we get $\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2}$. This result is mathematically rigorous, unlike Heisenberg's result, and conceptually the two are quite distinct. One refers to simultaneous measurements on a single system, whereas the other refers to variances of statistics for ensembles of measurements where only one operator is measured on each individual syste. Note that normally different experimental set-ups are required for measuring complementary observables.

These uncertainty relations form the basis for the idea of complementarity introduced by Bohr, as will be discussed later.

Mixed States

We have so far only considered pure states. A general state operator $\hat{\rho}$ can be defined by the conditions:

i) $\mathrm{tr}(\hat{\rho}) = 1$.

ii) $\langle u|\hat{\rho}|u\rangle \geq 0$ for all $|u\rangle \in \mathcal{H}$.

A state for which $\mathrm{tr}(\hat{\rho}^2) < 1$ is called a *mixed state*. For a finite dimensional system, the purity is bounded by $\frac{1}{\dim(\mathcal{H})} \leq \mathrm{tr}(\hat{\rho}^2) \leq 1$.

A mixed state can be 'created' in two ways: (i) as a convex combination of pure states due to classical ignorance (this is called *a proper mixture*), or (ii) by ignoring any quantum correlations between the system and some ancillary system, leaving only partial information about the system state (this is called *an improper mixture*).

Note that there is an infinite number of different convex combinations of pure states for the same mixed state (this is called *the ambiguity of mixtures*.)

Consequences of Classical Ignorance

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Generalized Quantum States from Classical Ignorance

Lecture 3 Jan 11, 2005

Imagine an imperfect device that yields $|\psi_1\rangle\langle\psi_1|$ with probability p_1, and a different state $|\psi_1\rangle\langle\psi_2|$ with probability $p_2 = 1-p_1$. To describe such a preparation we use a state operator of the form,

$$\hat{\rho} = p_1|\psi_1\rangle\langle\psi_1| + (1-p_1)|\psi_2\rangle\langle\psi_2| \tag{1}$$

, where we are assuming $|\psi_1\rangle \in \mathcal{H}$, $|\psi_2\rangle \in \mathcal{H}$ such that $\langle\psi_1|\psi_2\rangle \neq 0$ and $\mathrm{tr}\left(|\psi_{1,2}\rangle\langle\psi_{1,2}|\right) = 1$.

This state operator has the properties,

- $\mathrm{tr}(\hat{\rho}) = 1$
- $\langle u|\hat{\rho}|u\rangle \geq 0$ for all $|u\rangle \in \mathcal{H}$
- $\mathrm{tr}(\hat{\rho}^2) \leq 1$

If $\dim(\mathcal{H}) = d$, then

$$\frac{1}{d} \leq \mathrm{tr}(\hat{\rho}^2) \leq 1. \tag{2}$$

The maximally mixed state $\hat{\rho} = \frac{\hat{1}}{d}$ represents total ignorance about the system.

The continuous variable generalization is

$$\hat{\rho} = \int d\alpha\, p(\alpha)|\psi(\alpha)\rangle\langle\psi(\alpha)|. \tag{3}$$

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Generalized Quantum Transformations from Classical Ignorance

Now consider a (non-ideal) device implementing with probability p_1 a unitary $\hat{U}_1$, and with probability $1-p_1$ implementing $\hat{U}_2 \neq \hat{U}_1$. A linear map $\mathcal{E}(\hat{\rho})$ defined by

$$\mathcal{E}(\hat{\rho}) = p_1\hat{U}_1\hat{\rho}\hat{U}_1^\dagger + (1-p_1)\hat{U}_2\hat{\rho}\hat{U}_2^\dagger \equiv \hat{\rho}_f \tag{4}$$

is not unitary, but it is positive (i.e. it takes state operators to state operators).

In general, $\mathrm{tr}(\mathcal{E}(\hat{\rho})^2) \leq \mathrm{tr}(\hat{\rho}^2)$ with equality if $p_1 = 0$ or 1, that is, unitary operators preserve the purity, but these linear operators can only decrease the purity.

The continuous variable generalization is given by

$$\mathcal{E}(\hat{\rho}) = \int d\alpha \, p(\alpha) \hat{U}(\alpha) \hat{\rho} \hat{U}(\alpha)^\dagger. \tag{5}$$

Generalized Quantum Measurements from Classical Ignorance

Take some Hermitian operator $\hat{O} = \hat{O}^\dagger$. Then

$$\hat{O} = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k| \tag{6}$$

For some state $\hat{\rho}$, the probability of the outcomes are

$$\Pr(k) = \mathrm{tr}(\hat{\rho}|\lambda_k\rangle\langle\lambda_k|) = p_k \tag{7}$$

where the p_k can be estimated by the relative frequency of the number of outcomes k with respect to the total number of tests/experiments. With that information, one can calculate

$$\langle\hat{O}\rangle = \mathrm{tr}(\hat{O}\hat{\rho}) = \sum_k \lambda_k p_k, \tag{8}$$

but also the expectation of any operator decomposed by the same projectors, e.g.

$$\langle\hat{O}^2\rangle = \mathrm{tr}\left(\sum_k \lambda_k^2 |\lambda_k\rangle\langle\lambda_k|\hat{\rho}\right) = \sum_k \lambda_k^2 p_k. \tag{9}$$

So the crucial thing here is not the Hermition operator, but the projectors $\hat{P}_k = |\lambda_k\rangle\langle\lambda_k|$ over which it is decomposed, and their associated probabilities p_k. These projectors need not be rank one. In general, these measurements are called *projector valued measurements* (PVM), the term is due to von Neumann (1932).

Now consider a measurement device which implements a PVM $\{|\phi_i\rangle\langle\phi_i|\}$ with probability p_1, and the PVM $\{|\psi_i\rangle\langle\psi_i|\}$ with probability $(1-p_1)$, i.e.

$$\{\hat{E}_i\} = \{p_1|\phi_i\rangle\langle\phi_i| + (1-p_1)|\psi_i\rangle\langle\psi_i|\}, \tag{10}$$

which implies

$$\begin{aligned} p_i &= p_1 \mathrm{tr}(\hat{\rho}|\phi_i\rangle\langle\phi_i|) + (1-p_1)\mathrm{tr}(\hat{\rho}|\psi_i\rangle\langle\psi_i|) & (11)\\ &= \mathrm{tr}[\hat{\rho}(p_1|\phi_i\rangle\langle\phi_i| + (1-p_1)|\psi_i\rangle\langle\psi_i|)] & (12)\\ &= \mathrm{tr}[\hat{\rho}\hat{E}_i] & (13)\end{aligned}$$

and

$$\langle u|\hat{E}_i|u\rangle \geq 0 \text{ for all } |u\rangle \in \mathcal{H} \tag{14}$$

so the $\hat{E}_i$ are positive operators. Also, $\sum_i p_i = 1$ iff $\sum_i \hat{E}_i = 1$.

The $\hat{E}_i$ are called *positive operator valued measurements* (POVM). Note that in general $\hat{E}_i^2 \neq \hat{E}_i$, and that PVMs are a special case where $\hat{E}_i^2 = \hat{E}_i$.

Any operator $\hat{O}$ has a matrix representation in a basis $\{|\phi_i\rangle\}$. The matrix elements are given by $\langle\phi_i|\hat{O}|\phi_j\rangle = O_{ij}$ where i is the row number, and j is the column number.

Tensor Product Structure and Composite Systems

Say we have two systems S_A and S_B, and the total system $S = S_A + S_B$. The Hilbert space of the total system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and we use the convention that $|\psi_A\rangle \in \mathcal{H}_A$, $|\psi_B\rangle \in \mathcal{H}_B$ and $|\psi\rangle \in \mathcal{H}$.

Let $\{|\psi_{A(B),i}\rangle\}$ be a basis for $\mathcal{H}_{A(B)}$, so that, for example, $|\psi_A\rangle = \sum a_i |\psi_{A,i}\rangle$, where $a_i = \langle\phi_{A,i}|\psi_A\rangle$ and $|\psi\rangle = \sum_{ij} c_{i,j}|\phi_{A,i}\rangle \otimes |\phi_{B,j}\rangle$ so the $\{|\phi_{A,i}\rangle|\phi_{B,j}\rangle\}$ form a basis for $\mathcal{H}$.

An operator $\hat{A} \otimes \hat{B}$ acts as $\hat{A} \otimes \hat{B}|\psi\rangle = \sum_{ij} c_{ij}\hat{A}|\phi_{A,i}\rangle \otimes \hat{B}|\phi_{B,j}\rangle$.

Any operator $\hat{O}$ acting on $\mathcal{H}$ can be expressed $\hat{O} = \sum_k \hat{A}_k \otimes \hat{B}_k$.

Define $\hat{O} \equiv \hat{A} \otimes \hat{1}$ then $\langle\hat{O}\rangle = \mathrm{tr}[(\hat{A} \otimes \hat{1})|\psi\rangle\langle\psi|]$ We can define $\hat{\rho}_A = \mathrm{tr}_B|\psi\rangle\langle\psi|$ so that $\langle\hat{O}\rangle = \mathrm{tr}[\hat{\rho}_A\hat{A}]$.

How is the partial trace calculated exactly?

$$\hat{\rho}_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|) \tag{15}$$

$$= \sum_j \langle\phi_{B,j}|\psi\rangle\langle\psi|\phi_{B,j}\rangle \tag{16}$$

which uniquely defines a state operator over subsystem S_A.

Entanglement

A pure state $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is entangled (not factorable, not separable) iff it cannot be expressed as

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \tag{17}$$

There is a simple criterion for entanglement in pure states: $\mathrm{tr}(\hat{\rho}_A^2) < 1$. That is, if the reduced density matrix is pure, than the system is not entangled.

Another useful result is the Schmidt decomposition. For all $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there are bases $\{|\chi_{A,i}\rangle\}\{|\chi_{B,i}\rangle\}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively such that

$$|\psi\rangle = \sum_i \alpha_i |\chi_{A,i}\rangle \otimes |\chi_{B,i}\rangle \tag{18}$$

where $\alpha_i \geq 0$, $\sum_i \alpha_i^2 = 1$. $|\psi\rangle$ is entangled iff the number of non-zero α_i is greater than 1. Using the Schmidt decomposition it easy to show that

$$\mathrm{tr}(\hat{\rho}_A^2) = \mathrm{tr}(\hat{\rho}_B^2) \text{ if } |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B. \tag{19}$$

Generalized Postulates of Quantum Theory

Similar to the way we can obtain a mixed state on $\mathcal{H}_A$ from a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ by ignoring one of the subsystems (tracing over it), we can also obtain generalized transformations $\mathcal{E}(\hat{\rho}_{i,A}) = \hat{\rho}_{f,A}$ on $\mathcal{H}_A$ that are not unitary by tracing over the B part of a unitary operator $\hat{U}$ acting on $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. We can also obtain POVMs on $\mathcal{H}_A$ by tracing out the B component of a PVM acting on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Postulate I. The most general description of a system's configuration is given by a non-negative, unit trace, state operator $\hat{\rho}$. Rank-one projectors $\hat{\rho} = |\psi\rangle\langle\psi|$ are state of maximal knowledge.

Postulate II. Every measurement can be described by a set $\{\hat{E}_k\}$ of positive operators such that $\sum_k \hat{E}_k = 1$. The proability of the outcome labeled by k, for state $\hat{\rho}$, is given by $\Pr(k) = p_k = \mathrm{tr}(\hat{\rho}\hat{E}_k)$. Rank one PVMs correspond to maximal tests.

Postulate III. The most general description of a tranformation is a completely positive linear map $\mathcal{E}(\hat{\rho})$ which takes the set of positive operators to itself. A *completely positive* (CP) operator is an operator $\mathcal{E}$ such that $\mathcal{E}(\hat{\rho}_A \otimes \hat{1}) = \hat{\rho}_{AB}$ where both $\hat{\rho}_A \otimes \hat{1}$ and $\hat{\rho}_{AB}$ are both non-negative. Physically realizable operations are required to be CP – for example, the partial transpose is not a CP map.

While any mixed state can be represented by classical ignorance, there are general operations that cannot be represented by classical ignorance over unitary operations. Some general operations can only arise when an auxiliary quantum system has been traced over.

Sequential Measurements: von Neumann on the Compton Experiment

In the experiment where photons are scattered off electrons that are initially at rest (by Compton), von Neumann considered the scenario where there is a finite time difference Δt between the interception of e^- and the photon (the detection of these particles). There are 3 logically possible degrees of randomness:

1) The momentum of the e^- and the photon have uncorrelated dispersions
2) The momentum of e^- has a dispersion, but that momentum of the photon is in a fixed correlation with the momentum of the e^-
3) Both particles' momenta can be predicted (from the initial conditions) with certainty (no dispersion)

Experiments tell us that (2) is the actual case. This is the practical reason why von Neumann was forced to introduce the collapse postulate.

Projection Postulate (von Neumann, 1932)

von Neumann realized from the analysis of sequential measurements that two kinds of transformation were required in quantum mechanics:

Process 1. After observation/measurement of an outcome λ_k, the system is left in the eigenstate $|\lambda_k\rangle$ associated with the detected eigenvalue λ_k. We have the map,

$$\hat{\rho} \rightarrow |\lambda_k\rangle\langle\lambda_k|.$$

von Neumann called this process an essential randomness in nature, and he considered it grounds for abandoning the "principle of sufficient cause."

Process 2. The normal Schrödinger (unitary) evolution.

Interpretation according to Bohr, von Neumann, and Dirac

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Overview of the Significant Features of Quantum Theory

Wave-particle duality: coherent superpositions and interference effects due to wave-like properties for particles (e.g, two-slit experiment) and the particle-like behaviour of EM waves (e.g., photoelectric effect).

What is the physical meaning of a coherent superposition of distinct position eigenstates of an electron?

Invasive role of measurement: the necessity of state disturbance and the intrinsic limitations of the measurement process (e.g., Heisenberg microscope).

Indeterminism and uncertainty: Robertson's uncertainty principle (the unavoidable dispersion for non-commuting observables) and the loss of determinism (causality).

There are two possible explanations for the characteristic dispersions that are predicted by quantum theory: i) a fundamental randomness (stochasticity) in nature, or ii) the existence of additional (hidden) coordinates which fully determine the experimental outcomes. Next week we take this up by considering the EPR argument for incompleteness of the wavefunction, and then turn to a discussion of the (known) contraints on hidden variables (non-locality and contextuality).

The ambiguous ontology of quantum states: The quantum measurement problem for the orthodox interpretation arises from the assumption that quantum states provide a complete description of physical properties. A related problem is to understand the physical status of the projection postulate and an unambiguous set of rules for defining when it may be correctly applied.

The Measurement Process:

Projection postulate: Upon measurement of a non-degenerate observable $R = \sum \lambda_k |\phi_k\rangle\langle\phi_k|$, the following transformation is required:

$$\rho(t) \to \rho'(t) = \sum_k \langle\phi_k|\rho(t)|\phi_k\rangle \; |\phi_k\rangle\langle\phi_k| \tag{1}$$

This is von Neumann's process 1, also called the 'collapse postulate', or 'state reduction.'

Projection destroys the coherence of the state (across the eigenstates of R) and eliminates the possibility of observing interference effects from the prior phase-relations in subsequent experiments.

This entropy-increasing rule (representing a pure $\to$ mixed transition) applies when the actual outcome is 'ignored.'

If we wish to describe the state of a sub-ensemble which is post-selected based on a certain outcome (e.g., the outcome $\{\lambda_j\}$, then the quantum state operator for that sub-ensemble is,

$$\rho(t) \to \rho'(t) = |\phi_j\rangle\langle\phi_j|. \tag{2}$$

In particular, this rule applies *when describing an individual system* where outcome λ_j has been obtained.

The post-selected projection postulate can not be deduced from the unitary transformation law (von Neumann's 'process 2.') on either the system alone, or on the system combined with some other auxiliary systems.

In the most general measurement of an observable, the system is not left in an eigenstate. For example, when a photon's momentum is measured by absorption it is clear that the photon is not left in a momentum eigenstate - it is destroyed by the measurement. Pauli calls the ideal von Neumann measurements "measurements of the first kind." Measurements of the first kind, combined with post-selection, are an important means of "state preparation."

von Neumann and Dirac on the Measurement Process:

The projection postulate can not be deduced from the unitary dynamics:

“We have then answered the question as to what happens in the measurement of [an observable]. To be sure, the “how” remains unexplained for the present. This discontinuous transition from the wavefunction into one of [the eigenstates of the observable] is certainly not of the type described by the time dependent Schrdinger equation. This latter always results in a continuous change of [the wavefunction], in which the final result is uniquely determined and is dependent on [the wavefunction].”

von Neumann (1932/1955), p. 217

The projection postulate is demanded for consistency with experiment:

“From physical continuity, if we make a second measurement of the same dynamical variable immediately after the first, the result of the second measurement must be the same as that of the first.”

Dirac (1947)

von Neumann goes through a long analysis to show that the application of the projection postulate can be applied in a consistent way either to the system directly or to the system + apparatus, but insists that ultimately the postulate must be applied to describe the outcomes obtained by the act of observation:

"That is, we must always divide the world into two parts, the one being the observed system, the other the observer. In the former, we can follow up all physical processes (in principle at least) arbitrarily precisely. In the later, this is meaningless. The boundary between the two is arbitrary to a very large extent. ...That this boundary can be pushed arbitrarily deeply into the interior of the body of the actual observer is the content of the principle of the psycho-physical parallelism - but this does not change the fact that in each method of description the boundary must be put somewhere, if the method is not to proceed vacuously, i.e., if a comparison with experiment is to be possible. Indeed experience only makes statements of this type: an observer has make a certain (subjective) observation; and never any like this: a physical quantity has a certain value.

Now quantum mechanics describes the events which occur in the observed portions of the world, so long as they do not interact with the observing portion, with the aid of the process 2 [Schrodinger evolution], but as soon as such an interaction occurs, i.e., a measurement, it requires the application of the process 1 [projection postulate]. The dual form is therefore justified."

von Neumann (1932/1955), p. 418-419

What does it mean to interpret a theory?

The minimum interpretation that is required is to develop the most general set of rules that enable the theory to predict experimental outcomes. These operational bridge principles serve a purely epistemological role (from the Greek *episteme*, meaning 'knowledge').

Operationalism:

"In a strict sense, quantum theory is a set of rules alowing computation of probabilities for the outcomes of tests which follow specified preparations."

A. Peres (1995), p 13.

The idea of operationalism is to relate the elements of the mathematical formalism to the operations one can perform in the lab and to the outcomes one can observe. This level of interpretation fulfills the practical role of a physical theory as a means of predicting and controlling physical systems.

Operational interpretations are the minimum needed for the 'shut up and calculate' application of quantum theory.

"Ordinary quantum mechanics (as far as I know) is just fine FOR ALL PRACTICAL PURPOSES [FAPP]"

It should emphasized from this operational context that quantum theory can make at most only statistical predictions, i.e., statistics with finite dispersion for some observables.

But what is really going on?

In addition to the practical solution offered by operationalism, it is interesting and important to ask: what can we say about the underlying reality that is consistent with quantum theory. That is, we demand a set of *ontological* bridge principles (from the Greek *ontos*, meaning 'to be'). These principles, or correspondence rules, are needed also to develop the explanatory aspect of a physical theory, that is, to provide some explanation or story about the nature of the physical world.

The ontological bridge principles can also be practical in the sense that they can give intuition about how systems behave and about what results to expect when direct calculation is infeasible. Moreover, the answers can hopefully give insight into how to further develop and unify our physical theories (e.g., the problem of unifying quantum theory with gravity.)

Ontological question for quantum theory:

How do the mathematical elements of quantum theory correspond to physical properties of nature?

What is the physical status of quantum states, e.g, are they complete?

What constraints does quantum theory impose on the possibility of assigning objective properties to observables?

Is a fully deterministic description in terms of objective properties possible?

Interpretation of classical physics: ontology vs epistemology

A good example of an ontological theory is Hamiltonian mechanics: definite objective properties (q, p) may be assigned to a system and these properties evolve deterministically according to the canonical equations:

The state $(q(t), p(t))$ is complete in the sense that these values (combined with the dynamical laws) provide all the information that is needed to determine the physical state of the system at any time.

A good example of an epistemological classical theory is Liouville mechanics. Here we have a state $\rho(q(t), p(t), t)$ which is a probability density that evolves deterministically according to:

The state $\rho(q, p, t)$ represents (incomplete, subjective) knowledge about the system's physical properties.

There is a close correspondence between the Liouville state ρ and the quantum state ρ and Liouville mechanics and quantum mechanics.

1. The deterministic dynamical law is similar to the Schrodinger evolution. In fact, due to Koopman's theorem, the Liouville time-evolution may be represent by a unitary operator (sometimes called the Frobenius-Perron operator).
2. The outcomes upon measurement are (subjectively) random.
3. An independent transformation rule is required to account for the state after measurement: a classical collapse postulate is required to describe the state after measurement (this transformation is non-unitary).

The Copenhagen Interpretation:

Defining complementarity:

"*Complementarity:* any given application of classical concepts precludes the simultaneous use of other classical concepts which in a different connection are equally necessary for the elucidation of the phenomena."

Bohr (1934), p. 10

Causality and maybe also objective physical reality must be rejected:

"... a subsequent measurement to a certain degree deprives the information given by a previous measurement of its significance for predicting the future course of the phenomena. Obviously, these facts not only set a limit to the extent of the information obtainable by measurements, but they also set a limit to the meaning which we may attribute to such information. We meet here in a new light the old truth that in our description of nature the purpose is not to disclose the real essence of the phenomena but only to track down , so far as it is possible, relations between the manifold aspects of our experience."

Bohr (1934), p. 18

"An independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation."

Bohr (1928)

There is a necessary boundary between quantum and classical:

Bohr holds firm to the fact that although “atomic phenomena” must be described by quantum mechanics, our measuring devices must be described using classical physics:

“The experimental conditions can be varied in many ways but the point is that in each case we must be able to communicate to others what we have done and what we have learned, and that therefore the functioning of the measuring instruments must be described within the framework of classical physical ideas.”

Bohr (1934), p. 89

But by what criterion can we determine where the boundary arises? Bohr concedes there is none:

There is “the impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.”

Bohr (1949), p. 10

Were Bohr’s views informed by quantum mechanics, or imposed upon quantum mechanics?

“How closely the idea of complementarity was in accord with Bohr’s older philosophical ideas became apparent through an episode which took place ... on a sailing trip from Copenhagen to Svendborg on the Island Fyn... Bohr was full of the new interpretation of quantum theory, and as the boat took us full sail southward ... there was plenty of time to reflect philosophically on the nature of atomic theory. ... Finally, one of Bohr’s friends remarked drily, ‘But Niels, this is not really new, you said exactly the same ten years ago.’ ”

Heisenberg (1967)

The 'Orthodox' Dirac-von Neumann Interpretation:

The quantum state gives a complete description of physical properties:

"In this method of description, it is evident that everything which can be said about the state of a system must be derived from its wave function."

von Neumann (1932/1955), p. 196

The very limited conditions under which observables have actual, definite values are prescribed by the 'eigenvalue-eigenstate link' (term due to A. Fine, 1973):

"The expression that an observable 'has a particular value' for a particular state is permissible in quantum mechanics in the special case when a measurement of the observable is certain to lead to the particular value, so that the state is an eigenstate of the observable. In the general case we cannot speak of an observable having an value for a particular state, but we can speak of its having an average value for the state. We can go further and speak of the probability of its having any specified value for the state, meaning the probability of this specified value being obtained when one makes a measurement of the observable."

Dirac (1958)

The projection postulate introduces fundamental randomness and we must reject 'the principle of sufficient cause':

"This concept of quantum mechanics, which accepts its statistical expression as the actual for of the laws of nature, and which abandons the principle of causality, is the so-called statistical interpretation."

von Neumann (1932/1955), p. 210

The projection postulate represents a physical transformation or 'jump':

"When we measure a real dynamical variable, the disturbance involved in the act of measurement causes a jump in the state of the dynamical system. From physical continuity, if we make second measurement of the same dynamical variable immediately after the first, the result of the second measurement must be the same as that of the first. Thus after the first measurement has been made, there is no indeterminacy in the result of the second. Hence, after the first measurement has been made, the system is in an eigenstate of the dynamical variable, the eigenvalue it belongs to being equal to the result of the first measurement. This conclusion must still hold if the second measurement is not actually made. In this way we see that a measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured, the eigenvalue this eigenstate belongs to being equal to the result of the measurement."

Dirac (1958)

The concept of determinism must be abandoned:

"The question of causality could be put to a true test only in the atom, in the elementary processes themselves, and here everything in the present state of our knowledge militates against it. The only formal theory existing at the present time which orders and summarizes our experiences in this area in a half-way satisfactory manner, i.e., quantum mechanics, is in compelling logical contradiction with causality. Of course it would be an exaggeration to maintain that causality has thereby been done away with: quantum mechanics has, in its present form, several serious lacunae, and it may even be that it is false, although this latter possibility is highly unlikely, in the face of its startling capacity in the qualitative explanation of general problems, and in the quantitative calculation of special ones."

von Neumann (1932/1955), p. 327

Dissent from Einstein

On the Copenhagen Interpretation:

"The Heisenberg-Bohr tranquilizing philosophy - or religion? - is so delicately contrived that, for time being, it provides a gentle pillow for the true believer from which he cannot very easily be aroused. So let him lie there."

A. Einstein (1928)

Against the orthodox view that quantum mechanics provides a complete description of individual systems:

"One arrives at very implausible theoretical conceptions, if one attempts to maintain the thesis that the statistical quantum theory is in principle capable of producing a complete description of an individual physical system I am convinced that everyone who will take the trouble to carry through such reflections conscientiously will find himself finally driven to this interpretation of quantum-theoretical description (the ψ-function is to be understood as the description not of a single system but of an ensemble of systems) . . . There exists, however, a simple psychological reason for that fact that this most nearly obvious interpretation is being shunned. For if the statistical quantum theory does not pretend to describe the individual system (and its development in time) completely, it appears unavoidable to look elsewhere for a complete description of the individual system . . . Assuming the success of efforts to accomplish complete physical description, the statistical quantum theory would, within the framework of future physics, take an approximately analogous position to the statistical mechanics with the framework of classical mechanics. I am rather firmly convinced that the development of theoretical physics will be of that type; but the path will be lengthy and difficult. "

A. Einstein (1949)

The more complete analysis Einstein seeks "is *in principle* excluded."

Bohr (1949), p. 35

In reference to the Copenhagen viewpoint:

"To believe this is logically possible without contradiction; but it is so very contrary to my scientific instinct that I cannot forego the search for a more complete description."

Einstein (1936)

Have these issues been satisfactorily resolved ... ?

" I think that I can safely say that nobody understands quantum mechanics"

R. Feynman (Nobel Laureate, 1965)

The measurement problem a la Schrodinger:

"A cat is penned up in a steel chamber, along with the following diabolical device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny amount of radioactive substance, so small, that perhaps in the course of one hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if mean while no atom has decayed. The first atomic decay would have poisoned it. The ψ-function of the entire system would express this by having in it the living and the dead cat (pardon the expression) mixed or smeared out in equal parts.

Schrodinger (1935)

The measurement problem, more formally:

Consider an atom described by a pure state corresponding to a coherent superposition of moving along two distinct trajectories. We arrange so that both trajectories pass through a detector such that a macroscopic pointer is moved to the 'left' if the atom is on the 'up' trajectory and to the 'right' if the atom is on the 'down' trajectory. If we treat the problem quantum mechanically and demand faithful measurements we get,

$$(\alpha|\mathrm{up}\rangle + \beta|\mathrm{down}\rangle) \otimes |\mathrm{ready}\rangle \to \alpha|\mathrm{up}\rangle \otimes |\mathrm{left} > +\beta|\mathrm{down}\rangle \otimes |\mathrm{right}\rangle, \quad (3)$$

such that after the interaction the apparatus pointer is in an entangled superposition with the atom's trajectory.

The orthodox view:

On the orthodox view, an observable property is definite only if the system is in an eigenstate of the operator associated with the observable and a pure state offers a complete description of the physical properties of a system. Our classical experience tells us that the position of the macroscopic pointer should be a definite property. However, the pointer is not in a position eigenstate.

Once we 'observe' the macroscopic pointer, we must apply the projection postulate, then the system + pointer state collapses to a product of position eigenstates, e.g. $|\mathrm{down}\rangle \otimes |\mathrm{right}\rangle$, and each has a definite position property. (It should be emphasized here that such a reduction of the state can not be derived from the unitary evolution of the system + whatever other degrees of freedom, and hence consists of an independent process.)

Criticism of the orthodox view:

How and when does the collapse process occur? In the case of Schrodinger's cat, is the cat's life status in an undefined state until it is observed? Or is the cat's own observation enough to collapse the wavefunction?

[Added after lecture] This point is perhaps emphasized best by Bell (in 'Against Measurement', 1990): "What exactly qualifies some physical systems to play the role of 'measurer'? Was the wavefunction of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system ... with a Ph D?"

This problem of what constitutes an observation has led some to speculate that there is an infinite regress until we ultimately consider our personal direct observations, that is, we may have to resign ourselves to solipsism. This is illustrated by the dilemma of "Wigner's friend" (Wigner, 1962), in which Wigner imagines a friend observing the outcome of an experiment, such as the pointer position, and points out the unitary evolution implies that his friend's mental state is in an entangled superposition with the system + apparatus state, and asks whether the collapse should occur only when Wigner himself asks his friend what he has observed.

The main problem is: which physical systems should we describe with quantum concepts and which should we describe with classical concepts? Here the old problem for Bohr of where to draw the boundary between quantum and classical descriptions is brought to a point against the orthodox interpretation.

The projection postulate has a natural analog in Liouville mechanics when a (finite-precision) measurement is preformed. The required state update rule (after measurement) can not be described by the classical dynamical equation (e.g., it does not have a unitary representation). However, this is not conceptually problematic since the Liouville state is not presumed to be a physical (ontological) state with an objective status. Rather it has an epistemic status, and hence the transformation is not presumed to be a physical process. The classical projection corresponds to an update of the observer's information about the system.

For Einstein, the measurement problem is further evidence that the quantum description is not complete:

"The attempt to conceive the quantum-theoretical description as the complete description of the individual systems leads to unnatural theoretical interpretations, which become immediately unnecessary if one accepts the interpretation that the description refers to ensembles of systems and not to individual systems."

Einstein (1949)

Extra Quotes:

"...It is in principle impossible ...to formulate the basic concepts of quantum mechanics without using classical mechanics."

Landau and Lifschitz, p. 2

"...Thus quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet as the same time requires this limiting case for its own formulation."

Landau and Lifschitz, p. 3

The debate with Einstein concerned "whether the renunciation of a causal mode of description of atomic processes ...should be regarded as a temporary departure from ideals to be ultimately revived or whether we are faced with an irrevocable step towards obtaining the proper harmony between analysis and synthesis of physical phenomena."

Bohr (1949), p. 2

"Now from my conversations with Einstein I have seen that he takes exception to the assumption, essential to quantum mechanics, that state of a system is defined only by specification of an experimental arrangement. Einstein wants to know nothing of this. If one were able to measure with sufficient accuracy, this would of course be as true for small macroscopic spheres as for electrons. It is, of course, demonstrable by specifying though experiments, and I presume that you have mentioned and discussed some of these in your correspondence with Einstein, But Einstein has the philosophical prejudice that (for macroscopic bodies) a state (termed 'real') can be defined 'objectively' under any circumstances, that is, without specification of the experimental arrangement used to examine the system (of the macro-bodies), or to which the system is being 'subjected'. It seems to me that the discussion with Einstein can be reduced to this hypothesis of his, which I have called the idea (or the 'ideal') of the 'detached observer.' But to me and other representatives of quantum mechanics, it seems that there is sufficient experimental and theoretical evidence against the practicability o this ideal."

W. Pauli - letter to ? (1954)

"First, it is inherently entirely correct that the measurement of the related process of the subjective perception is a new entity relative to the physical environment and is not reducible to the latter. Indeed, subjective perception leads us into the intellectual inner life of the individual, which is extra-observational by its every nature) since it must be take for granted by any conceivable observation or experiment). Nevertheless, it is a fundamental requirement of the scientific viewpoint - the so-called principle of the psycho-physical parallelism - that it must be possible so to describe the extra-physical process of the subjective perception as if it were in reality in the physical world - i.e., to assign to its parts equivalent physical processes in the objective environment, in ordinary space. "

von Neumann (1955), p. 418-419

"While the means of observation (experimental arrangements an apparatus, records such as spot on photographic plates) have still to be described in the usual 'common language supplemented with the terminology of classical physics,' the atomic 'objects' used in the theoretical interpretation of the 'phenomena' cannot any longer be described 'in a unique way by conventional physical attributes.' Those 'ambiguous' objects used in the description of nature have an obviously symbolic character."

W. Pauli (1948)

" the development of a state according to 1. is statistical, while according to 2. it is causal."

von Neumann (1955), p. 356

"...one should expect that 2. would suffice to describe the intervention caused by a measurement. Indeed, a physical intervention can be nothing else than the temporary insertion of a certain energy coupling into the observed system, i.e., the introduction of an appropriate time dependency of H (prescribed by the observer). Why then do we need the special process 1. for the measurement?"

von Neumann (1955), p. 352

Problems for the Orthodox Interpretation

Lecturer: Joseph Emerson

What does it mean to interpret a theory?

Operational vs Ontological Bridge-Principles

Operational Bridge-Principles

Operational bridge-principles are operational rules that relate elements of the formalism to measurements that may be performed.

These rules provide the adequate information to use quantum theory in the lab, i.e., to explain experimental outcomes (observations).

Operational rules do not give insight into the nature of the underlying physical reality of the systems described by quantum theory.

An important example of an operational bridge-principle in quantum theory is the Born rule, which tells us the relative frequency (probability) with which outcome k is observed given the same measurement repeated on an ensemble of identically prepared systems:

$$\mathrm{Prob}(k|\rho) = \mathrm{tr}(\rho|\phi_k\rangle\langle\phi_k|)$$

Ontological Bridge-Principles

Ontological bridge-principles are a set of correspondence rules that relate elements of the mathematical formalism to elements of physical reality.

Bohr's "Copenhagen Interpretation" can be considered ontological, in spite of his denial of the meaningfulness of making statements about an independent reality,

> "An independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation."

because he also insists that deducing additional information about what properties a system may have is impossible in principle.

> The more complete analysis Einstein seeks "is *in principle* excluded."

The Orthodox (Dirac-von Neumann) Interpretation

Postulate 1. Eigenvalue-eigenstate link.

> An observable has a determinate value *if and only if* the state is an eigenstate of that observable.

This is an ontological bridge-principle: it tells what properties a system possesses *independent of observation*.

The eigenvalue-eigenstate link implies that the *quantum state provides a complete description of a system's objective physical properties*, or put more boldly, of the objective elements of physical reality.

The completeness assumption implies that the unavoidable non-vanishing dispersion of outcomes for some observables (as demanded by Robertson's uncertainty principle) is due to a fundamental randomness (or stochasticity) in nature.

Note that von Neumann was well aware that the existence of additional "hidden coordinates" provided another possible explanation for the non-vanishing dispersion associated with quantum states. However, he rejected this possibility because of an 'impossibility proof' that he devised against the existence of dispersion-free assignments to all observables (via hidden variables). However, von Neumann's proof was discredited much later by Bohm (1952), who constructed an explicit hidden variable model for quantum theory. As we will see in the next lecture, the problem with von Neumann's proof was explicitly exposed by Bell (1966), who showed that one of von Neumann's assumptions on hidden variables was unreasonably strong in the sense that it constituted a 'no go' theorem against only a trivial classical of hidden variable extensions of quantum theory.

Postulate 2. The projection postulate.

> After an ideal measurement of an observable, the system state is transformed into [i.e., must be updated to] the eigenstate associated with the eigenvalue observed.

This postulate, also known as the collapse of the wavefunction, is *operationally demanded* for consistency with experiments involving sequential (ideal) measurements.

The eigenvalue-eigenstate link implies that the 'projection' is *a physical process* since it involves a transformation of the system's physical properties.

In contrast, if we reject the eigenvalue-eigenstate link, and if we reject that the quantum state is a complete description of a system's physical properties, then the 'projection' after measurement *does not correspond to not a physical process*.

The projection is then just an 'update rule' involving a change to an abstract theoretical construct, such as a (subjective) probability assignment, which must be updated when new information is obtained.

Projection Postulate with and without Post-Selection:

Consider the ideal measurement of a non-degenerate observable $R = \sum \lambda_k |\phi_k\rangle\langle\phi_k|$.

If the measurement outcome is ignored, then the following transformation,

$$\rho(t) \rightarrow \rho'(t) = \sum_k \langle\phi_k|\rho(t)|\phi_k\rangle\ |\phi_k\rangle\langle\phi_k|. \tag{1}$$

is required to describe the state after measurement.

If, on the other hand, the outcome is recorded, then consistency with subsequent measurements demands the following transformation:

$$\rho(t) \rightarrow \rho'(t) = |\phi_k\rangle\langle\phi_k|. \tag{2}$$

Can we model the projection postulate describing an ideal measurement using a unitary transformation?

While the transformation (1) may be modeled by a unitary acting on the system combined with an additional system when the additional system is ignored, the transformation (2) may not be modeled in this way and thus must be an independent process.

Proof that projection (under post-selection) is not a unitary process

Conceptually it is clear that unitary evolution evolves any given state to a fixed final state. This is deterministic (in the sense of reproducible). In contrast, collapse is fundamentally stochastic: applying the same measurement to the same preparation produces different (apparently random) final states (depending on the outcome).

Is it possible that the final state outcome is not random but dependent on the quantum state associated with some additional degrees of freedom, and the whole process may be described by a unitary transformation?

Consider an atom described by a pure state corresponding to a coherent superposition of moving along two distinct trajectories. We arrange so that both trajectories pass through a detector such that a macroscopic pointer is moved to the 'left' if the atom is on the 'up' trajectory and to the 'right' if the atom is on the 'down' trajectory. We want to model the measurement process with a unitary transformation and for complete generality we extend the quantum system to include additional degrees of freedom denoted by a state $|\chi\rangle$. If we demand *faithful measurements* this means that we must have, for any $|\chi\rangle$,

$$\begin{aligned} U|\mathrm{up}\rangle \otimes |\mathrm{ready}\rangle \otimes |\chi\rangle &= |\mathrm{up}\rangle \otimes |\mathrm{left}\rangle \otimes |\chi'\rangle \\ U|\mathrm{down}\rangle \otimes |\mathrm{ready}\rangle \otimes |\chi\rangle &= |\mathrm{down}\rangle \otimes |\mathrm{right}\rangle \otimes |\chi''\rangle \end{aligned}$$

where $|\chi'\rangle$ and $|\chi''\rangle$ are allowed to be independent of $|\chi\rangle$.

Now if we prepare a coherent superposition over atomic trajectories, and allow for both possible outcomes, then by linearity it follows that, for any χ,

$$\begin{aligned} U(\alpha|\mathrm{up}\rangle + \beta|\mathrm{down}\rangle) \otimes |\mathrm{ready}\rangle \otimes |\chi\rangle &= \alpha|\mathrm{up}\rangle \otimes |\mathrm{left}\rangle \otimes |\chi'\rangle \\ &\quad +\beta|\mathrm{down}\rangle \otimes |\mathrm{right}\rangle \otimes |\chi''\rangle \end{aligned}$$

so it is impossible that after the interaction the state is driven to one or the other outcome. Hence the transformation (2) can not be modeled by a unitary transformation.

Proof that projection (without post-selection) can be represented by a unitary process

If we describe only the state of the system and pointer, then we must take a partial trace over the ancillary degrees of freedom represent by $|\chi\rangle$. This partial trace produces the following state (after measurement),

$$\begin{aligned} \rho &= |\alpha|^2\ |\mathrm{up}\rangle\langle\mathrm{up}|\ +\ |\beta|^2\ |\mathrm{down}\rangle\langle\mathrm{down}| + \\ &\quad \alpha\beta^*\ \langle\chi''|\chi'\rangle\ |\mathrm{up}\rangle\langle\mathrm{down}| + \alpha^*\beta\ \langle\chi'|\chi''\rangle\ |\mathrm{down}\rangle\langle\mathrm{up}| \end{aligned}$$

If the ancillary states are orthogonal $\langle\chi'|\chi''\rangle = 0$, then we recover the projection postulate describing the final state when the outcome is ignored or unknown:

$$\rho = |\alpha|^2\ |\mathrm{up}\rangle\langle\mathrm{up}|\ +\ |\beta|^2\ |\mathrm{down}\rangle\langle\mathrm{down}|.$$

Hence the projection transformation (1) (without post-selection) can indeed be modeled by a unitary transformation.

This important process is called “decoherence.” It shows us that the non-classical features of a coherent superposition, such as interference, are eliminated if the system or apparatus is allowed to interact with ancillary degrees of freedom which are either ignored or unknown.

If one takes into account the microscopic degrees of freedom associated with ever-present “environment” systems, such as dust particles, or the cosmic microwave background, these can be included via the ancillary states $|\chi\rangle$. Since these environment systems are unavoidably interacting with all macroscopic systems, such as pointers on measurement devices, these environment systems will continuously couples to the pointer states, producing coherently entangled superpositions as described above. Moreover, it is reasonable to infer that the environment states couple to macroscopically distinct pointer positions will become orthogonal after interacting with (reflecting off of) the macroscopically distinct pointer states. Hence for generic macroscopic systems, where the environment system state are not recorded, the pointer + atom states will reduce to the mixture described above.

These consideration explain why it is difficult to observe interference effects *in practice* in macroscopic systems possessing many degrees of freedom. Recall that one never observes interference effects with an individual system. Interference effects can only be observed by collecting statistics over an ensemble of identically prepared systems. Moreover, the interference fringes that may be observed in the probability distributions appear on an increasingly ‘short-wavelength’ as the mass of the system increases. Thus finer and finer resolution is required as we approach the macroscopic world. So there are many more reasons than just decoherence available for explaining why we are not readily observing ‘interference effects’ in our everyday observations of the world.

Does decoherence solve the measurement problem?

Can we interpret the final mixed state as an ordinary classical mixture of the two possible pointer positions?

A first problem with this approach is the "ambiguity of mixtures" (discussed last week). While the state describing the final pointer state may be interpreted as a classical mixture of the two possible pointer positions, this is a non-unique decomposition of the mixed state. It is possible also to re-express the state as a mixture of two very non-classical states that have nothing to do with well-defined pointer positions. This is called the "preferred basis problem".

A second problem is that the 'total system' is still in a pure state (coherent entangled superposition). The state for the combined system clearly does not allow the assignment of definite position properties for elements of the combined system (consisting of the pointer and the atom and the ancillary degrees of freedom). Is it self-consistent to deny definite properties for the combined system while asserting definite properties for a subsystem?

Can we conclude that, because of decoherence, there is no longer a conflict between the orthodox interpretation and the existence of definite position property for the macroscopic pointer?

There is a common misconception that the practical consideration of the environment, which generically produces the decoherence transition of the form described as transformation (1) earlier (because of the ubiquitous coupling of macroscopic devices to these microscopic environment degrees of freedom), solves the measurement problem. However, as shown above, decoherence can not explain the occurrence of transformation (2), which lies at the heart of the measurement problem.

Recall that the eigenvalue-eigenstate link of the orthodox interpretation tells us that definite properties for the positions of the atom and pointer should be assigned if and only if the combined state is a factorable state of the form,

$$|\psi\rangle = |\mathrm{up}\rangle \otimes |\mathrm{left}\rangle.$$

A mixed state obtained by partial tracing over the environment (or over the environment and the atom) is not in this form and therefore can not be assigned a definite property.

Hence, even if decoherence effects are taken into account, the orthodox interpretation still needs the post-selected projection postulate (transformation (2)) to explain the existence of macroscopic facts, and the measurement problem remains unsolved.

Contemporary Interpretations

A consistent description of macroscopic facts requires either expanding upon or rejecting the interpretative postulates of the orthodox interpretation.

'Dynamical collapse' interpretations specify the exact conditions under which collapse occurs by adding a non-linear term to the Schrodinger equation. Strictly speaking these interpretations are actual modifications of the mathematical formalism and not just interpretations in the sense of specifying ontological bridge-principles. P. Pearle will describe spontaneous collapse models to us in March.

The many-worlds interpretation developed by Everett (1957) rejects the projection postulate and imagines reality dividing into alternate but equally valid branches. This interpretation will be described to us next week by D. Wallace.

In many contemporary interpretations the effects of decoherence play a pivotal role in defining the ontology.

> One example is the "existential interpretation" advocated by W. Zurek (1993), which is a variation of the many-worlds interpretation.
>
> Another example is the "decoherent/consistent histories" interpretation, developed by R. Griffiths (1984) and extended by Gell-Mann and Hartle (1990), which will be described to us by R. Griffiths in March.

Last but not least we have interpretations which reject the assumption that quantum states provide a complete specification of system's properties.

> On the one hand there is the statistical interpretation, developed by Ballentine (1970), which, following Einstein, merely reject the completeness assumption and emphasizes the statistical/epistemic nature of the quantum state. This perspective will be explained by Ballentine in February, and further developed by myself and Rob Spekkens in March.
>
> On the other hand there are interpretations which seek to explicitly identify the additional 'hidden variables' needed for a complete specification of the system's properties. The most important example of this kind of interpretation is the de Broglie-Bohm (1927/1952) pilot wave theory, which will be introduced to us by S. Goldstein in February, and further elaborated by A. Valentini in March.

On Thursday we will spend our last introductory lecture discussing the constraints on hidden variables.

Incompleteness and Constraints on Hidden Variables

Lecturer: Joseph Emerson

Are quantum states complete?

'Einstein Attacks Quantum Theory. Scientist and Two Colleagues Find It Is Not "Complete" Even Though "Correct."' New York Times, May 4th, 1935.

The EPR Argument against Completeness

"Can quantum-mechanical description of reality be considered complete?"

A. Eintein, B. Podolosky, and N. Rosen, Physical Review 47, 777 (1935).

For EPR, a necessary condition for the completeness of a theory is:

(i) "Every element of physical reality must have a counterpart in the physical theory."

This is not a sufficient condition for completeness: there may be other criteria that must be satisfied.

For EPR, a sufficient condition for the physical reality of a quantity is:

(ii) "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

This is not a necessary condition: there may be other ways to identify whether a physical quantity is real.

Following from the fact that quantum mechanical states did not permit simultaneous specification of definite properties for non-commuting observables, EPR deduced that: "either (1) *the quantum-mechanical description of reality given by the wave function is not complete*, or (ii) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.*"

Both of these alternative inferences were appreciated already by von Neumann, who explicitly endorsed (2) and rejected (1), presumably due to his 'no go' theorem for hidden variables. The EPR argument concludes that (1) must be endorsed and (2) rejected.

EPR considered a system of two particles initially interacting such that they are produced in a joint eigenstate of their relative position and total linear momentum. Here we will consider a simpler system involving a two spin-1/2 particles (devised by Bohm (1951)) which illustrates the same features.

Consider two particles arranged to interact such that they are described by the singlet-state,

$$\psi = \frac{1}{\sqrt{2}} \left(|+\rangle_1 \otimes |-\rangle_2 \; - \; |-\rangle_1 \otimes |+\rangle_2 \right).$$

This state has zero total angular momentum, so the spin of the first particle (system S_1) is anti-correlated with the spin of the second particle (system S_2).

Assume that after the state preparation *the two particles no longer interact*.

Observe that if measurement of particle 1, along, say, the z-axis, yields $+\hbar/2$ then measurement of particle 2 (along the same z-axis) must yield $-\hbar/2$, and vice versa. Similarly, if we measure instead S_x for particle 1, then we can predict with certainty the outcome of an S_x measurement for particle 2.

Hence we can predict with certainty the outcomes of measurements of either S_x or S_z of the second particle ‘without in any way disturbing the second system”.

In accordance with the EPR criterion of reality, there must therefore be elements of reality corresponding to both S_x and S_z for the second particle.

Hence option (2) is negated. Since the two options are considered mutually exclusive and jointly exhaustive possibilities, EPR were *forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete*.

Bohm emphasized that the EPR argument relied on the additional assumptions that (iii) each element of physical reality must have a precisely defined counterpart in the mathematical theory [a stronger condition then the one EPR acknowledged] and (iv) the world can correctly be analyzed in terms of distinct and separately existing ‘elements of reality’.

Bohm (1951, pp. 622-623) somehow concluded from his analysis that hidden variables were nonetheless impossible: “We can now use some of the results of the analysis of the paradox of [EPR] to help prove that quantum theory is inconsistent with the assumption of hidden causal variables . . . [Arguing from the apparent conflict with the uncertainty principle] . . . We conclude that no theory of mechanically determined hidden variables can lead to all of the results of the quantum theory.”

One year later, Bohm published his causal hidden variable interpretation (1952): “A Suggested Interpretation of the Quantum Theory in Term of ‘Hidden’ Variables”

Additional observations:

If the collapse of the wave function is a physical process (as it is on the assumption that the wave function is complete), then the collapse must be an instantaneous change of physical properties throughout space. Recall from von Neumann's analysis of the Compton experiment that the process 1 transformation, with post-selection, is operationally demanded so that measurements of commuting operators yield consistent outcomes, even if the measurements are carried out 'simultaneously' in vastly separated spatial locations.

EPR did not linguistically distinguish, as we have, between the bare mathematical formalism (the abstract theory), and the set of bridge principles that specify correspondence rules between the elements of the mathematics and the elements of physical reality (the interpretation the theory). EPR referred to the combination of both the mathematical formalism and the orthodox interpretation as 'quantum theory.' So it is unclear if the argument for incompleteness is meant to imply the mathematical formalism itself, or merely the interpretive correspondence rules.

Bohr's response to EPR:

"The finite interaction between object and measuring agencies conditioned by the very existence of the quantum of action entails - because of the impossibility of controlling the reaction of the object on the measuring instruments if these are to serve their purpose - the necessity of a final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical reality . . . [While there is] no question of a mechanical disturbance of the system under investigation . . . there is essentially the question of an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system."

Bohr "Quantum Mechanics and Physical Reality" (1935)

"Recapitulating, the impossibility of subdividing the individual quantum effects and separating a behavior of the objects from their interaction with the measuring instruments serving to define the conditions under which the phenomena appear implies and ambiguity in assigning conventional attributes to atomic objects which calls for a reconsideration of our attitude towards the problem of physical explanation, in this novel situation, even the old question of an ultimate determinacy of natural phenomena has lost its conceptual basis, and it is against this background that the viewpoint of complementarity presents itself as a rational generalization of the very ideal of causality."

Bohr (1948)

Non-locality

The EPR argument presumes (implicitly) a notion of *separability*, i.e., that separately existing elements of reality may be attributed to each system, and a notion of *independence*, i.e., that it is possible to arrange that the elements of reality of one system can not be influenced by the elements of reality of another system. The assumption of independence can seemingly be well motivated by the 'locality' guaranteed by special relativity. Einstein later characterized this 'locality' assumption as follows (1949):

> "The real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former."

This 'Einstein locality' assumption was put to a direct test by John Bell (1964) who devised a celebrated inequality that any local, realistic theory must satisfy, as we will see in a moment.

Incompleteness and hidden variables:

The conclusion that quantum mechanics is incomplete suggests that the quantum mechanical description may be supplemented by additional parameters, or 'hidden variables', in order to recover a (more) complete description.

As noted earlier, the possibility of hidden variables was considered and rejected by von Neumann, who produced an 'impossibility proof' based on a number of assumptions.

First, the hidden variables assignments were required to completely specify experimental outcomes (i.e., to produce dispersion-free states) for all Hermitian operators;

Second, the unique hidden value assignment to each operator was required to be one of the operator's eigenvalues;

Third, for any Hermitian operator $\hat{C} = a\hat{A} + b\hat{B}$ defined by a linear combination of arbitrary (e.g., non-commuting) Hermitian operators, the hidden value assignment for $\hat{C}$ was required to satisfy the same linear combination of the hidden value assignments for the operators $\hat{A}$ and $\hat{B}$.

However, the third assumption is incompatible with the first and second assumptions. Consider the Pauli operator defined by,

$$\sigma_n = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y).$$

The eigenvalues of all three Pauli operators are ± 1, but clearly, the eigenvalues of σ_n can not be expressed as any of the linear combinations,

$$\frac{(\pm 1 \pm 1)}{\sqrt{2}}.$$

von Neumann's third assumption is generally considered unjustified (even "silly" - by Mermin (1993)), since it imposes constraints on the hidden value assignments for incompatible experimental arrangements. His assumption appears to be inspired by the fact that this relation holds for quantum mechanical expectation values ($\langle C\rangle = \langle A\rangle + \langle B\rangle$) and maybe also the fact that it is expected in a trivial hidden variable model in which measurements of spin reveal the components of a pre-existing angular momentum vector.

If we drop the third assumption then hidden variable models can be, and indeed have been, constructed - see Bell (1966) for the complete analysis and a simple example. The most celebrated example is the de Broglie-Bohm hidden variable model (1927/1952), which has an explicit non-local character. This feature of the de Broglie-Bohm model interpretation and the role of locality in the EPR-Bohm argument led Bell to ask: is non-locality a necessary feature of any hidden variable theory reproducing the predictions of quantum theory? The answer is yes (Bell (1964)).

Bell's Theorem:

Bell considered a restriction on the correlations that can be exhibited between two systems in the EPR-Bohm set-up allowing for the fact that the outcomes could be determined by an arbitrary class of (deterministic) hidden variables.

Consider two spatially separated spin systems each subjected to measurement along directions a and b respectively. The results of the measurements, denoted A and B, can depend on arbitrary parameters (hidden variables) collectively denoted λ, and can take on the values $|A| \leq 1$ and $|B| \leq 1$. The outcome can of course depend on the local setting, but, by *assuming Einstein locality*, is not allowed to depend on the setting of the distant instrument. Hence $A = A(a, \lambda)$ and $B = B(b, \lambda)$ are allowed but $A = A(a, b, \lambda)$ and $B = B(a, b, \lambda)$ are excluded by the locality assumption.

The uncontrolled parameters are subject to an arbitrary probability density $\rho(\lambda)$, where,

$$\rho(\lambda) \geq 0, \qquad \int \rho(\lambda) d\lambda = 1,$$

and hence we can define correlations of the form:

$$C(a, b) = \int A(a, \lambda) B(b, \lambda) \rho(\lambda) d\lambda$$

Each detector is allowed to have two *independently selected* settings $\{a, a'\}$ and $\{b, b'\}$. From these assumptions we can deduce Bell's inequality:

$$|C(a, b) - C(a, b')| + |C(a', b') + C(a', b)| \leq 2$$

A quantum mechanical system satisfying Bell's assumption consists of two spin-1/2 particles (or generic two-level systems) in the singlet state,

$$\psi = \frac{1}{\sqrt{2}} \left(|+\rangle_A \otimes |-\rangle_B \; - \; |-\rangle_A \otimes |+\rangle_B \right).$$

Define the observable correlation as,

$$C(\mathbf{a}, \mathbf{b}) = (\mathbf{2}/\boldsymbol{\hbar})^{\mathbf{2}} \, \langle \psi | \mathbf{a} \cdot \mathbf{S_A} \otimes \mathbf{b} \cdot \mathbf{S_B} | \psi \rangle.$$

Define $\cos\theta_{\mathbf{a},\mathbf{b}} \equiv \mathbf{a} \cdot \mathbf{b}$, then,

$$C(\mathbf{a}, \mathbf{b}) = -\cos\theta_{\mathbf{a},\mathbf{b}}$$

Choosing $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}'$ to be four co-planar vectors with $\mathbf{a}$ and $\mathbf{b}$ parallel and $\phi \equiv \theta_{\mathbf{a},\mathbf{b}'} = \theta_{\mathbf{a}',\mathbf{b}}$, then the Bell inequality demands,

$$|1 + 2\cos(\phi) - \cos(2\phi)| \leq 2$$

but this is violated for a wide range of ϕ.

Exercise 1 (Assignment 2): Derive Bell's inequality. Calculate $C(a, b)$ for the singlet state. Plot $|1 + 2\cos(\phi) - \cos(2\phi)|$ to show the range and degree of violation of Bell's inequality.

Observations:

The kind of locality that is violated by quantum mechanics is called *weak locality* because the violation does not permit 'super-luminal signaling.' That is, only a random sequence of outcomes are obtained at either location and the non-local correlations (on their own) can not be used to communicate information to the distant party. In contrast, a theory violating *strong locality* would allow the possibility of super-luminal signaling, e.g., rigid body mechanics.

Bell's argument relies also on an assumption of determinism: the outcomes are determined by the hidden variable $A = A(a, \lambda)$ and $B = B(b, \lambda)$. However, Clauser, Horne, Shimony, and Holt (1969), and Clauser and Horne (1974) developed inequalities (that assume local hidden variables and which are violated by quantum mechanics) that do not also rely on the determinism assumption - only a probabilistic dependence on the hidden variables is presumed.

It is worth noting that Bell-type inequalities presume that the detector settings at the two separated locations may be selected independently, for example, by the 'free will' of the experimenters, or by some sufficiently pseudo-random function. Ultimately, in a fully deterministic conception of the world, all events could be traced back to a common cause, and are never truly independent.

Are hidden variables non-local or is quantum mechanics non-local?

Bell-type inequalities tell us that any hidden variable models reproducing quantum theory must be non-local. However, if we reject hidden variable models, and assume instead that the quantum state is a complete description of a system's physical properties, then the EPR analysis shows (implicitly) that because quantum states must be updated (collapsed) non-locally, it follows that physical properties of the world are exhibiting non-locality. *So whether one accepts or rejects that quantum states are a complete description, one is forced to accept non-locality.*Some authors (Stapp, 1985, 1988) even conclude that sequences of experimental outcomes which violate Bell-type inequalities *imply* that non-locality is a feature of the world, rather than just a feature of quantum mechanics. In this sense the violation of Bell's inequalities should not be viewed as a reason to reject hidden variables, but as a required constraint on hidden variable models.

The quantum mechanical violation of Bell-type inequalities has been demonstrated experimentally, in a number of distinct experiments. Some of the most important early experiments were performed by Alain Aspect and co-workers (1981,1982). Aspect will be giving us lectures on Bell-type inequalities and the experimental tests on March 15th and 17th.

Hidden Variable Assignments and the Contextuality Constraint

Consider again the idea that quantum statistics arise from incomplete knowledge of pre-existing values for all observables. On this assumption, for any observable A we wish to assign some pre-existing value $v_\psi(A) \in R$, where the subscript reminds us that the assignment will in general depend on the preparation.

What kinds of requirements should the value assignments satisfy?

Earlier we rejected von Neumann's requirement that the value assignments individually satisfy the same relations as the statistical averages (give some preparation).

A more innocent requirement is that any function of the value assigned to a commuting operators $\{A, B, C \dots\}$ should be equal to the value of the function of the operators

$$f(v_\psi(A), v_\psi(B), v_\psi(C), \dots) = v_\psi(f(A, B, C, \dots)).$$

This requirement implies the properties:

$$\text{(i)} \;\; v_\psi(A+B) = v_\psi(A) + v_\psi(B) \quad \text{if} \quad [A, B] = 0$$

$$\text{(ii)} \quad v_\psi(AB) = v_\psi(A) v_\psi(B) \quad \text{if} \quad [A, B] = 0$$

$$\text{(iii)} \quad v_\psi(\mathbb{1}) = 1.$$

Recall the spectral decomposition of a non-degenerate observable:
$A = \sum_k \lambda_k \hat{P}_k$ where the projector $\hat{P}_k = |\phi_k\rangle\langle\phi_k|$ satisfies $\hat{P}_k^2 = \hat{P}_k$.
These properties allow us to deduce that projectors must be assigned values according to:

$$v_\psi(P_k) \in \{1, 0\}.$$

Also note that the value assignment to a general Hermitian operator must be one of its eigenvalues:

$$v_\psi(A) \in \{\lambda_k\}$$

Bell-Kochen-Specker theorem: If the Hilbert space dimension is greater than two, a consistent value assignment constrained by these properties is not possible.

This result was obtained by Bell (1966) and, independently, by Kochen and Specker (1967). Bell's proof assumed value assignments to a continuum of projectors, whereas the Kochen-Specker theorem required only a finite set (actually, 117 of them). The approach of the KS-proof was to consider sets of commuting triads which share a single vector in common, e.g., although $[A, B] = [B, C] = 0$, it does not follow that $[A, C] \neq 0$.

A simpler proof, requiring a 4-dimensional Hilbert space is due to Mermin (1993).

The consequence of the Bell-KS-type theorems is expressed by saying that value assignments to quantum mechanical observables are *contextual*. This means that the value assigned to any observable must depend on the specification of which other commuting observables are being assigned values along with it.

Bell's (self-)Criticism

"[we have] tacitly assumed that the measurement of an observable must yield the same value independently of what other measurements must be made simultaneously." Since some observables in each set (row or column in Mermin's proof, triads in the KS proof) do not commute with the additional observables in the other, the measurements of each complete set are incompatible. "These different possibilities require different experimental arrangements; there is no a priori reason to believe that the results . . . should be the same."

Bell (1966)

Bohr's Prescience

"[The] measuring instruments . . . serve to define the conditions under which the phenomena appear."

Bohr (1949)

Assigned Reading for Next Week's Lectures:
Many Worlds Interpretation by D. Wallace:

"Everett and Structure", quant-ph/0107144.

"Quantum Probability from Subjective Uncertainty", quant-ph/0312157.

Anyone particularly keen (all of you right?) should also look at:

Lev Vaidman's encyclopedia article "The Many-Worlds Interpretation of Quantum Mechanics", Stanford Encyclopedia of Philosophy, available at http://www.tau.ac.il/ vaidman/mwi/mw2.html.

Adrian Kent, "Against many-worlds interpretations", gr-qc/9703089

If you want a better understanding of decoherence read:

W. Zurek, quant-ph/0306072.